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Traveling-wave method for solving the modified nonlinear Schrödinger equation describing soliton propagation along optical fibers

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We give a traveling-wave method for obtaining exact solutions of the modified nonlinear Schrödinger equation $iu_t+\varepsilon u_{xx}+2p|u|^2u+2iq(|u|^2u)_x=0$, describing the propagation of light pulses in optical fibers, where u represents a normalized complex amplitude of a pulse envelope, t is the normalized distance along a fiber, and x is the normalized time within the frame of reference moving along the fiber at the group velocity. With the help of the "potential function" we obtained by this method, we find a family of solutions that are finite everywhere, particularly including periodic solutions expressed in terms of Jacobi elliptic functions, stationary periodic solutions, and "algebraic" soliton solutions. Compared with previous work [D. Mihalache and N. C. Panoiu, J. Math. Phys. **33**, 2323 (1992)] in which two kinds of the simplest solution were given, the physical meaning of the integration constants in the potential function we give is clearer and more easily fixed with the initial parameters of the light pulse.

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I. INTRODUCTION

Optical solitons in fibers are pulses that propagate without any change in pulse shape or intensity. Because of their remarkable stability properties, optical solitons are now at the center of an active research field of nonlinear wave propagation in optical fiber. This research field started with the results obtained by Hasegawa and Tappert [1], which showed that, under appropriate combinations of pulse shape and intensity, the effects of the intensity-dependent refractive index of the fiber exactly compensate for the pulse-spreading effects of group-velocity dispersion. For the negative group velocity dispersion or anomalous dispersion regime, the fundamental soliton is called a bright pulse, and the propagation of these bright solitons has been studied intensively and verified experimentally [2]. For the positive group velocity dispersion or normal-dispersion regime, the theory [1] and numerical simulations [3] predict that the solitons are dark pulses (i.e., a dip occurs at the center of the pulse). The generation of the dark solitons in monomode optical fibers was also demonstrated [4]. We also mention the works of several active research groups in the field of the theory of pulse propagation in optical fibers in both the picosecond and femtosecond regimes [5–23].

The propagation of the optical pulse in a monomode optical fiber exhibiting Kerr-law nonlinearities is described well by the following modified nonlinear Schrödinger equation (MNLSE) [24–27]:

$$iu_t+\varepsilon u_{xx}+2p|u|^2u+2iq(|u|^2u)_x=0,\quad \varepsilon=\pm 1\ , \tag{1.1}$$

where u represents a normalized complex amplitude of the pulse envelope, t is the normalized distance along the fiber, and x is the normalized time within the frame of the reference moving along the fiber at the group velocity. The subscripts indicate partial derivatives. This equation describes the propagation of picosecond light pulses ($p=1$, $q=0$, and $\varepsilon=1$ for the anomalous-dispersion regime and $\varepsilon=-1$ for the normal-dispersion regime), and in certain circumstances this MNLSE can be used to describe the propagation of femtosecond pulses in optical fibers (see Refs. [21] and [22]). The case when $p\neq 0$ and $q=0$ was studied by the inverse-scattering method [28–34] and an ansatz method [35–37]. Exact analytical solutions for the higher-order nonlinear Schrödinger equation were found by using Lie group theory [38]. In Ref. [39], by using the inversing-scattering method, the solution of the derivative nonlinear Schrödinger equation (DNLSE), i.e., $p=0$, $\varepsilon=1$, and $q=\pm\frac{1}{2}$, was given. We mentioned that the MNLSE (1.1) can be reduced to the following DNLSE:

$$iv_t+v_{xx}+i(|v|^2v)_x=0 \tag{1.2}$$

via the transformation

$$u(x,t)=\left[\frac{2q}{\sqrt{\varepsilon}}\right]^{-1/2}\exp\left[i\left[\frac{p}{q}x-\frac{\varepsilon p^2}{q^2}t\right]\right]\times v\left[\frac{x}{\sqrt{\varepsilon}}-\frac{2p\sqrt{\varepsilon}}{q}t,t\right]\ . \tag{1.3}$$

A method for solving (1.1) in the picosecond region ($q=0$) was found recently [35,40,41]. This method was based on a linear relationship between the real and imaginary parts of the complex pulse envelope $u(x,t)$ with coefficients which depend only on the "time" variable t.

Exact analytical solutions for MNLSE (1.1) were given in Ref. [42]. This method is based on a series of transformations of the pulse amplitude $|u(x,t)|$ and the variables t and x. We notice that more integration constants were

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contained in solutions given in Ref. [42], and that the physical meaning of them was ambiguous. It is especially very difficult to fix these integration constants with the initial conditions of the pulse.

In this paper, using the traveling-wave method, we give a family of analytical solutions of the MNLSE (1.1) in the fs region. It will be shown that all of the integration constants in these solutions can be fixed easily with the initial parameters of the pulse, and that the physical meaning of them are very clear.

II. THE DESCRIPTION OF THE METHOD

For solving Eq. (1.1) we put the unknown function $u(x,t)$ in the traveling-wave form

$$u(x,t)=r(\xi)\exp\{i[\theta(\xi)+\omega t]\}\ , \tag{2.1}$$

where $\xi=x-vt$, v is the wave velocity, $r(\xi)$ is real amplitude of the wave, $\theta(\xi)$ is the parameter of the phase modification, and ω is the frequency of the wave oscillation. By putting $u(x,t)$ in Eq. (2.1) into Eq. (1.1), and taking the real and imaginary parts of Eq. (1.1), we obtain the following system of ordinary differential equations:

$$\varepsilon(r\theta''+2r'\theta')-vr'+6qr^2r'=0\ , \tag{2.2}$$

$$(vr-2qr^3)\theta'-\varepsilon r\theta'^2+\varepsilon r''+2pr^3-\omega r=0\ , \tag{2.3}$$

where r' is the first order derivative of r to ξ, r'' is the second one, θ' is the first order derivative of θ to ξ, and θ'' is the second one. There is the following first integration in Eq. (2.2):

$$\varepsilon r^2\theta'-vr^2/2+3qr^4/2=A/2\ , \tag{2.4}$$

i.e.,

$$\theta'=(v+A/r^2-3qr^2)/2\varepsilon\ , \tag{2.4$'$}$$

where A is an integration constant. Setting Eq. (2.4′) into Eq. (2.3), we have

$$r''+3q^2r^5/4-(qv-2\varepsilon p)r^3+(v^2/4+qA/2-\varepsilon\omega)r-A^2/4r^3=0\ . \tag{2.5}$$

There is the following first integration in Eq. (2.5):

$$r'^2/2+q^2r^6/8-(qv-2\varepsilon p)r^4/4+(v^2/4+qA/2-\varepsilon\omega)r^2/2+A^2/8r^2+B=0\ , \tag{2.6}$$

i.e.,

$$s'^2+q^2s^4-2(qv-2\varepsilon p)s^3+(v^2+2qA-4\varepsilon\omega)s^2+8Bs-A^2=0\ , \tag{2.6$'$}$$

where B is the second integration constant, and $s=r^2$.

In order to analyze different kinds of solutions of nonlinear differential equation (2.6′), we rewrite it as follows:

$$s'^2+v(s)=0\ , \tag{2.7}$$

where

$$v(s)=q^2s^4-2(qv-2\varepsilon p)s^3+(v^2+2qA-4\varepsilon\omega)s^2+8Bs-A^2 \tag{2.8}$$

is a "potential function." Therefore, Eq. (2.7) describes the dynamics of a particle with a total energy zero and a mass 2 in the potential well $v(s)$. This potential is useful for predicting the behavior of different kinds of solutions of Eq. (2.6). The integration constants A and B in Eqs. (2.4) and in (2.6) which effect the shape of the potential well can easily be fixed with the initial conditions $r(\xi=0)$ and $\theta'(\xi=0)$.

III. RESULTS AND DISCUSSION

In the following we show how to obtain the general solution of the Eq. (2.7) and give all kinds of possible solutions. For different kinds of root distributions of the polynomial $v(s)$, there are different kinds of solutions for Eq. (2.7). Because $s=r^2\geq 0$ and all of the coefficients in $v(s)$ are real, we will discuss the real solution of the Eq. (2.7) only.

(1) In the case when all four roots of the polynomial $v(s)$ are real.

(i) All four roots are equal. In this case "potential function" $v(s)$ can be rewritten as

$$v(s)=q^2(s-\alpha)^4\ . \tag{3.1}$$

For this case $v(s)$ is always positive or zero, so there is no real solution in Eq. (2.7).

(ii) There is a single root and a triple root.

In this case "potential function" $v(s)$ can be rewritten as

$$v(s)=q^2(s-\alpha)^3(s-\beta)\ . \tag{3.2}$$

$v(s)<0$, when $\alpha<s<\beta$, if $\alpha<\beta$, or when $\beta<s<\alpha$, if $\beta<\alpha$. So there is a real solution for Eq. (2.7):

$$s(\xi)=\frac{\beta+\alpha[c+q(\alpha-\beta)\xi/2]^2}{1+[c+q(\alpha-\beta)\xi/2]^2}\ , \tag{3.3}$$

where c is an integration constant. This is an "algebraic" soliton solution. Under the condition of $\alpha,\beta\geq 0$, if $\beta>\alpha$ it represents a bright soliton; if $\beta<\alpha$, it represents a dark soliton.

For simplicity, we let $c=0$. Setting (3.3) into (2.4′), if $\alpha\neq 0$ we have

$$\theta=\theta_0+\frac{\varepsilon}{2}\left[v+\frac{A}{\alpha}-3q\alpha\right]\xi-\frac{A\varepsilon}{q\alpha\sqrt{\alpha\beta}}\arctan\left[\frac{q}{2}-(\beta-\alpha)\frac{\sqrt{\alpha}}{\sqrt{\beta}}\xi\right]-3\varepsilon\arctan[q(\beta-\alpha)\xi]\ . \tag{3.4}$$

In the case of $\alpha=0$, $s(\xi)$ in Eq. (3.3) and $\theta(\xi)$ in Eq. (3.4) are simplified, respectively, as

$$s(\xi)=\frac{\beta}{1+q^2\beta^2\xi^2/4}\ , \tag{3.5}$$

$$\theta(\xi)=\theta_0+\varepsilon(v+A/\beta)\xi+\varepsilon Aq^2\beta\xi^3/24-3\varepsilon\arctan(q\beta\xi/2)\ . \tag{3.6}$$

If we let $A=0$, $\varepsilon=1$, $q=\frac{1}{2}$, and $\beta=16\Delta^2$, then we have

$$u(x,t)=\frac{4\Delta}{[1+16\Delta^2(x-vt)^2]^{1/2}}\times\exp\{i[\theta_0+v(x-vt)/2-3\arctan[4\Delta^2(x-vt)]+\omega t]\}\ , \tag{3.7}$$

where θ_0 is another integration constant.

Now we discuss the possibility of dark soliton existence in normal dispersion ($\varepsilon=-1$). For simplicity let $\beta=0$ and $c=0$ in Eq. (3.2). We have

$$v(s)=q^2s(s-\alpha)^3\ . \tag{3.2$'$}$$

Comparing (3.2′) with (2.8), we have

$$2(vq+2p)/q^2=3\alpha>0\ , \tag{3.8a}$$

$$(v^2+4\omega)/q^2=3\alpha^2>0\ , \tag{3.8b}$$

$$8B/q^2=-3\alpha^3<0\ , \tag{3.8c}$$

where we have set $A=0$ and $\varepsilon=-1$ in Eq. (2.8). It can be seen from Eq. (2.6) that if the initial peak power and the group velocity of the pulse are as appropriate as $B<0$ in Eq. (2.6), which satisfies Eqs. (3.8a) and (3.8b), the pulse can evolve into a dark soliton.

(iii) There is a double root $s=\alpha$ and two single root $s=\beta$ and γ. In this case "potential function" $v(s)$ can be rewritten as

$$v(s)=q^2(s-\alpha)^2(s-\beta)(s-\gamma),\quad \beta>\gamma\ . \tag{3.9}$$

Because $v(s)<0$, if $\beta>s>\gamma$, there is a real solution for Eq. (2.7).

(a) If $\alpha>\beta$ or $\alpha<\gamma$, $s(\xi)$ takes the following form:

$$s(\xi)=\frac{\beta+\gamma[(\beta-\alpha)/(\gamma-\alpha)]\tan^2[\sqrt{(\gamma-\alpha)(\beta-\alpha)}(c+q\xi)/2]}{1+[(\beta-\alpha)/(\gamma-\alpha)]\tan^2[\sqrt{(\gamma-\alpha)(\beta-\alpha)}(c+q\xi)/2]}\ . \tag{3.10}$$

where $\gamma<s(\xi)<\beta$, when α and $\beta\geq 0$. This represents a period soliton.

(b) If $\gamma<\alpha<\beta$, $s(\xi)$ takes the form

$$s(\xi)=\frac{\beta+\gamma[(\beta-\alpha)/(\alpha-\gamma)]\tanh^2[\sqrt{(\alpha-\gamma)(\beta-\alpha)}(c+q\xi)/2]}{1+[(\beta-\alpha)/(\alpha-\gamma)]\tanh^2[\sqrt{(\alpha-\gamma)(\beta-\alpha)}(c+q\xi)/2]}\ . \tag{3.11}$$

It represents a bright soliton. Especially when $\alpha=0$, we obtain

$$s(\xi)=\frac{\beta\,\mathrm{sech}^2[\sqrt{-\beta\gamma}(c+q\xi)/2]}{1-(\beta/\gamma)\tanh^2[\sqrt{-\beta\gamma}(c+q\xi)/2]}\ . \tag{3.12}$$

In this case

$$v(s)=q^2s^2(s-\beta)(s-\gamma)\ . \tag{3.13}$$

Comparing it with Eq. (2.8), we have

$$\beta\gamma=(v^2-4\varepsilon\omega+2qA)/q^2\ , \tag{3.14a}$$

$$\beta+\gamma=(qv-2\varepsilon p)/q^2\ , \tag{3.14b}$$

$$A=0\ , \tag{3.14c}$$

$$B=0\ . \tag{3.14d}$$

They show that only when $\varepsilon=1$ does the solution Eq. (3.12) exist in Eq. (2.7).

Letting $q\rightarrow 0$, in the condition of Eq. (3.14c), Eq. (3.14d) and $\varepsilon=1$, the soliton solution (3.12) reduces to the following:

$$s(\xi)=\frac{4\omega-v^2}{4p}\,\mathrm{sech}^2[\sqrt{4\omega-v^2}(c'+\xi)/2]\ , \tag{3.15}$$

which we can obtain from Eq. (2.7) directly by setting $q=0$, $A=0$, and $B=0$ in Eq. (2.8). It represents a single-soliton traveling-wave solution of the NLSE:

$$iu_t+u_{xx}+2p|u|^2u=0\ . \tag{3.16}$$

(iv) There are two double roots $s=\alpha$ and β.

In this case "potential function" $v(s)$ can be rewritten as

$$v(s)=q^2(s-\alpha)^2(s-\beta)^2\ . \tag{3.17}$$

Because there is always $v(s)>0$ for any s value, it is impossible for a real solution for Eq. (2.7) to exist.

(v) There are two pairs of equal by opposite roots $s=\pm\alpha$ and $\pm\beta$.

In this case the "potential function" $v(s)$ can be rewritten as

$$v(s)=q^2(s^2-\alpha^2)(s^2-\beta^2),\quad \alpha>\beta\ . \tag{3.18}$$

Obviously, when $\beta<s<\alpha$ there is a real solution [43] in Eq. (2.7):

$$s(\xi)=\beta\,\mathrm{nd}(c+q\alpha\xi,k) \tag{3.19}$$

where c is an integration constant, and

$$\mathrm{nd}(x,k)=1/\sqrt{1-k^2\mathrm{sn}^2(x)} \tag{3.20}$$

is the Jacobi elliptic function, and in which

$$k^2=1-\beta^2/\alpha^2\ . \tag{3.21}$$

(vi) There are a pair of equal but opposite roots $s=\pm\alpha$, and two single roots $s=\beta$ and γ.

"Potential function" $v(s)$ can be written as

$$v(s)=q^2(s^2-\alpha^2)(s-\beta)(s-\gamma)\ . \tag{3.22}$$

For solving Eq. (2.8), in this case, we set

$$\lambda=[(\beta\gamma+\alpha^2)+\sqrt{(\gamma^2-\alpha^2)(\beta^2-\alpha^2)}]/(\beta+\gamma)\ , \quad (3.23)$$

$$\mu=[(\beta\gamma+\alpha^2)-\sqrt{(\gamma^2-\alpha^2)(\beta^2-\alpha^2)}]/(\beta+\gamma)\ , \quad (3.24)$$

$$b_1=\frac{2[(\beta\gamma+\alpha^2)-\sqrt{(\gamma^2-\alpha^2)(\beta^2-\alpha^2)}]}{\sqrt{(\gamma^2-\alpha^2)(\beta^2-\alpha^2)}}\ , \quad (3.25)$$

$$c_1=-\frac{2[(\beta\gamma+\alpha^2)+\sqrt{(\gamma^2-\alpha^2)(\beta^2-\alpha^2)}]}{\sqrt{(\gamma^2-\alpha^2)(\beta^2-\alpha^2)}}\ , \quad (3.26)$$

$$b_2=-\frac{[(\gamma^2-\alpha^2)+(\beta^2-\alpha^2)+2\sqrt{(\gamma^2-\alpha^2)(\beta^2-\alpha^2)}]}{\sqrt{(\gamma^2-\alpha^2)(\beta^2-\alpha^2)}}\ , \quad (3.27)$$

$$c_2=\frac{[(\gamma^2-\alpha^2)+(\beta^2-\alpha^2)-2\sqrt{(\gamma^2-\alpha^2)(\beta^2-\alpha^2)}]}{\sqrt{(\gamma^2-\alpha^2)(\beta^2-\alpha^2)}}\ , \quad (3.28)$$

$$T=(s-\lambda)/(s-\mu)\ , \quad (3.29)$$

$$s=(\lambda-\mu T)/(1-T)\ , \quad (3.30)$$

so we have

$$v(s)=q^2(s-\mu)^2(b_1T^2+c_1)(b_2T^2+c_2)\ . \quad (3.31)$$

(a) When $\beta^2,\ \gamma^2>\alpha^2$,

$$b_2=-\frac{(\sqrt{\beta^2-\alpha^2}+\sqrt{\gamma^2-\alpha^2})^2}{\sqrt{(\gamma^2-\alpha^2)(\beta^2-\alpha^2)}}<0\ , \quad (3.32)$$

$$c_2=\frac{(\sqrt{\beta^2-\alpha^2}-\sqrt{\gamma^2-\alpha^2})^2}{\sqrt{(\gamma^2-\alpha^2)(\beta^2-\alpha^2)}}>0\ , \quad (3.33)$$

(I) $\beta\gamma>0$.

$$b_1>0,\quad c_1<0\ ,$$
$$v(s)=q^2b_1b_2(s-\mu)^4(T^2-a^2)(T^2-b^2)\ , \quad (3.34)$$

where $a^2=-c_1/b_1$, $b^2=-c_2/b_2$, and

$$ds/\sqrt{-v(s)}=dT/[q\sqrt{-b_1b_2}\sqrt{(T^2-a^2)(T^2-b^2)}]\ , \quad (3.35)$$

where we have supposed $T>a$, $T>b$. Substituting it into Eq. (2.7) and using Eq. (3.30), we have

$$T(\xi)=b\ \mathrm{sn}(c+aq\xi,k)\ , \quad (3.36)$$

$$s(\xi)=[\lambda-\mu b\ \mathrm{sn}(c+aq\xi,k)]/[1-b\ \mathrm{sn}(c+aq\xi,k)]\ , \quad (3.37)$$

in which c is an integration constant, and

$$k^2=b^2/a^2\ . \quad (3.38)$$

(II) $\beta\gamma<0$.

$$b_1<0,\quad c_1>0\ ,$$
$$v(s)=q^2b_1b_2(s-\mu)^2(T^2-a^2)(T^2-b^2)\ , \quad (3.39)$$

where $a^2=-c_1/b_1$, $b^2=-c_2/b_2$, and

$$ds/\sqrt{-v(s)}=dT/[q\sqrt{b_1b_2}\sqrt{(T^2-a^2)(b^2-T^2)}]\ , \quad (3.40)$$

where we have supposed $b>T>a$. Substituting it into Eq. (2.7) and using Eq. (3.30), we have

$$T(\xi)=1/[a\ \mathrm{dn}(c+bq\xi,k)]\ , \quad (3.41)$$

$$s(\xi)=[a\lambda\ \mathrm{dn}(c+bq\xi,k)-\mu]/[a\ \mathrm{dn}(c+bq\xi,k)-1]\ . \quad (3.42)$$

c is an integration constant, and

$$k^2=1-a^2/b^2\ . \quad (3.43)$$

(a) When $\beta^2,\gamma^2<\alpha^2$,

$$b_2=\frac{(\sqrt{\beta^2-\alpha^2}-\sqrt{\gamma^2-\alpha^2})^2}{\sqrt{(\gamma^2-\alpha^2)(\beta^2-\alpha^2)}}>0\ , \quad (3.44)$$

$$c_2=-\frac{(\sqrt{\beta^2-\alpha^2}+\sqrt{\gamma^2-\alpha^2})^2}{\sqrt{(\gamma^2-\alpha^2)(\beta^2-\alpha^2)}}<0\ . \quad (3.45)$$

(I) $\beta\gamma>0$.

$$b_1>0,\quad c_1<0\ ,$$
$$v(s)=q^2b_1b_2(s-\mu)^2(T^2-a^2)(T^2-b^2)\ , \quad (3.46)$$

where $a^2=-c_1/b_1$, $b^2=-c_2/b_2$, and

$$ds/\sqrt{-v(s)}=dT/[q\sqrt{b_1b_2}\sqrt{(T^2-a^2)(b^2-T^2)}]\ , \quad (3.47)$$

where we have supposed $b>T>a$. The solution of Eq. (2.7) is the same as Eq. (3.42).

(II) $\beta\gamma<0$.

$$b_1<0,\quad c_1>0\ ,$$
$$v(s)=q^2b_1b_2(s-\mu)^2(T^2-a^2)(T^2-b^2)\ , \quad (3.48)$$

where $a^2=-c_1/b_1$, $b^2=-c_2/b_2$, and

$$ds/\sqrt{-v(s)}=dT/[q\sqrt{b_1b_2}\sqrt{(T^2-a^2)(T^2-b^2)}]\ , \quad (3.49)$$

where we have supposed $T>a, T>a$. The solution is the same as Eq. (3.37).

(vii) All four roots are different. In this case $v(s)$ can be rewritten as

$$v(s)=q^2(s-\alpha)(s-\beta)(s-\gamma)(s-\delta),\quad \alpha>\beta>\gamma>\delta\ . \quad (3.50)$$

When $\alpha>s>\beta$ or $\gamma>s>\delta$, $v(s)$ is integtive, and Eq. (2.7) contain real solutions which can be represented as follows.

(a) $\alpha>S>\beta$.

$$s(\xi)=\frac{\beta(\alpha-\gamma)-\gamma(\alpha-\beta)sn^2(c+q\sqrt{(\alpha-\gamma)(\beta-\delta)}\xi/2,k)}{(\alpha-\gamma)-(\beta-\gamma)sn^2(c+q\sqrt{(\alpha-\gamma)(\beta-\delta)}\xi/2,k)}\ . \quad (3.51)$$

(b) $\gamma > s > \delta$.

$$s(\xi)=\frac{\delta(\alpha-\gamma)+\alpha(\gamma-\delta)\mathrm{sn}^2(c+q\sqrt{(\alpha-\gamma)(\beta-\delta)}\xi/2,k)}{(\alpha-\gamma)+(\gamma-\delta)\mathrm{sn}^2(c+q\sqrt{(\alpha-\gamma)(\beta-\delta)}\xi/2,k)} , \quad (3.52)$$

in which c is an integration constant, and

$$k^2=(\alpha-\beta)(\gamma-\delta)/(\alpha-\gamma)(\beta-\delta) . \quad (3.53)$$

(2) In the case when there are two real roots and a pair of conjugate complex roots,

$$v(s)=q^2(s^2+\alpha^2)(s-\beta)(s-\gamma), \quad \beta>\gamma . \quad (3.54)$$

(i) $\beta=\gamma$

$$v(s)=q^2(s^2+\alpha^2)(s-\beta)^2 . \quad (3.55)$$

In this case, it is impossible for $v(s)<0$. So there is no real solution for Eq. (2.7).

(ii) $\gamma=-\beta$.

$$v(s)=q^2(s^2+\alpha^2)(s^2-\beta^2) . \quad (3.56)$$

When $s<\beta, v(s)<0$ and there is a real solution for Eq. (2.7),

$$s(\xi)=\frac{\alpha\beta}{\sqrt{\alpha^2+\beta^2}}\frac{\mathrm{sn}^2(c+q\xi\sqrt{\alpha^2+\beta^2},k)}{\sqrt{1-k^2\mathrm{sn}^2(c+q\xi\sqrt{\alpha^2+\beta^2},k)}} , \quad (3.57)$$

in which

$$k^2=\beta^2/(\alpha^2+\beta^2) . \quad (3.58)$$

(iii) $\gamma\neq\beta$, supposing $\beta>\gamma$.

For solving Eq. (2.7) in this case, let

$$\lambda=\frac{\beta\gamma-\alpha^2-\sqrt{(\beta\gamma-\alpha^2)^2+4\alpha^2(\beta+\gamma)^2}}{2(\beta+\gamma)} , \quad (3.59)$$

$$\mu=\frac{\beta\gamma-\alpha^2+\sqrt{(\beta\gamma-\alpha^2)^2+4\alpha^2(\beta+\gamma)^2}}{2(\beta+\gamma)} , \quad (3.60)$$

$$b_1=\frac{\beta\gamma-\alpha^2-\sqrt{(\beta\gamma-\alpha^2)^2+4\alpha^2(\beta+\gamma)^2}}{2\sqrt{(\beta\gamma-\alpha^2)^2+4\alpha^2(\beta+\gamma)^2}}<0 , \quad (3.61)$$

$$c_1=\frac{\beta\gamma-\alpha^2+\sqrt{(\beta\gamma-\alpha^2)^2+4\alpha^2(\beta+\gamma)^2}}{2\sqrt{(\beta\gamma-\alpha^2)^2 4\alpha^2(\beta+\gamma)^2}}>0 , \quad (3.62)$$

$$b_2=\frac{\beta\gamma-\alpha^2-\sqrt{(\beta\gamma-\alpha^2)^2+4\alpha^2(\beta+\gamma)^2}-2(\beta+\gamma)^2}{2\sqrt{(\beta\gamma-\alpha^2)^2+4\alpha^2(\beta+\gamma)^2}} , \quad (3.63)$$

$$c_2=\frac{\beta\gamma-\alpha^2+\sqrt{(\beta\gamma-\alpha^2)^2+4\alpha^2(\beta+\gamma)^2}-2(\beta+\gamma)^2}{2\sqrt{(\beta\gamma-\alpha^2)^2+4\alpha^2(\beta+\gamma)^2}} , \quad (3.64)$$

$$T=(s-\lambda)/(s-\mu) ,$$

so we have

$$v(s)=q^2b_1b_2(s-\mu)^4(T^2+c_1/b_1)(T^2+c_2/b_2) . \quad (3.65)$$

(a) For $c_2b_2>0$, let

$$a^2=-c_1/b_1, b^2=c_2/b_2 ,$$

$$s(\xi)=\frac{\lambda-a\mu\mathrm{nc}[c+q(\lambda-\mu)\sqrt{b_1b_1(a^2+b^2)}\xi,k]}{1-a\mathrm{nc}[c+q(\lambda-\mu)\sqrt{b_1b_1(a^2+b^2)}\xi,k]} , \quad (3.66)$$

where

$$k^2=a^2/(a^2+b^2) ,$$
$$\mathrm{nc}(x,k)=1/\sqrt{1-\mathrm{sn}^2(x,k)} . \quad (3.67)$$

(b) For $c_2b_2<0$, let

$$a^2=-c_1/b_1, \quad b^2=-c_2/b_2 , \quad (3.68)$$

$$s(\xi)=\frac{\lambda-b\mu\mathrm{nd}[c+q(\lambda-\mu)\sqrt{b_1b_1(a^2+b^2)}\xi,k]}{1-b\mathrm{nd}[c+q(\lambda-\mu)\sqrt{b_1b_1(a^2+b^2)}\xi,k]} , \quad (3.69)$$

where

$$k^2=1-b^2/a^2 , \quad (3.70)$$

in which we have supposed $a>b$.

(3) In the case when there are two pairs of conjugate complex roots,

$$v(s)=q^2(s^2+\alpha^2)(s^2+\beta^2) . \quad (3.71)$$

In this case, it is impossible for $v(s)<0$, so there is no real solution for Eq. (2.7).

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